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On bandwidth and edgesum for the composition of two graphs

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Abstract

The composition of two graphs G and H, written $G[H]$, is the graph with vertex set $V(G) \times V(H)$ and with (u_1, v_1) adjacent to (u_2, v_2) if either u_1 is adjacent to u_2 in G or $u_1 = u_2$ and v_1 is adjacent to v_2 in H. In this paper, we investigate the bandwidth problem for the composition of two graphs and obtain the bandwidth for several classes of graphs of the forms $K_n[G]$ and $K_{1,n}[G]$. We also provide optimal numberings which solve the graph edgesum problem for $K_n[P_m]$ and $K_n[C_m]$.

1. Introduction

Definition 1. Given two graphs G and H, the *product* (or *cartesian product*) of G and H, written $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) if u_1 is adjacent to u_2 in G and $v_1 = v_2$ or $u_1 = u_2$ and v_1 is adjacent to v_2 in H. For $n \geqslant 3$ the *product* of graphs $G_1, G_2, \ldots, G_n$ is defined inductively to be $G_1 \times G_2 \times \cdots \times G_n = (G_1 \times G_2 \times \cdots \times G_{n-1}) \times G_n$.

Definition 2. Let $G = (V, E)$ be a graph on n vertices. A 1–1 mapping $f: V \to \{1, 2, \ldots, n\}$ is called a *proper numbering* of G. The *bandwidth of a proper numbering* f of G, denoted $B_f(G)$, is the number

$$\max\{|f(u) - f(v)|: uv \in E(G)\}.$$

Definition 3. The *bandwidth of a graph* G, denoted $B(G)$, is the number

$$\min\{B_f(G): f \text{ is a proper numbering of } G\}.$$

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Definition 4. The *edgesum of a graph* G, denoted $s(G)$, is the number

$$\min\left\{\sum_{uv\in E(G)} |f(u)-f(v)| : f \text{ is a proper numbering of } G\right\}.$$

Several other terms have been used to denote what is here called *edgesum.* These include *bandwidth sum* [1], *minimum sum* [4], and *optimal linear arrangement* [8]. The term *edgesum* was used in [12].

Bandwidth on graphs, and the analogous problem of bandwidth on matrices, has been studied since the early 1950s (see [1]). The decision problem corresponding to finding the bandwidth of an arbitrary graph was shown to be NP-complete by Papadimitriou [18]. Garey et al. [7] showed that the problem is NP-complete even for trees of maximum degree 3. Harper [11] established the bandwidth for hypercubes (or n-cubes). The bandwidths for $P_n \times P_m$ and $P_n \times C_m$ were found by Chvátalová [5], [6]. Liu et al. ([15, 16]) solved a number of bandwidth problems relating to graph sums.

The edgesum problem was first investigated by Harper [10], who found the edgesum for the n-cube, Q_n. Lindsey [13] generalized Harper's result to the product of m complete graphs $K_{n1} \times K_{n2} \times \cdots \times K_{nm}$. Chung [3] established the most efficient known algorithm to achieve edgesum for complete binary trees. Recent results include establishing edgesum for $n \times n$ arrays [17], for a number of graph sums including $K_{n,m}$ [19], and for $P_n[P_m]$ [14]. The general problem of determining the edgesum for an arbitrary graph was shown to be NP-complete by Garey et al. [9].

In this paper, we investigate bandwidth and edgesum for the composition of two graphs. We first list some useful results:

Proposition 1 (Chinn et al. [1]). *For any n graphs* $G_1, G_2, \ldots, G_n$,

$$B(G_1 \times G_2 \times \cdots \times G_n) \leqslant \min\{B(G_i)\cdot \prod_{i\neq j} |V(G_j)| : i = 1,2,\ldots,n\}.$$

Proposition 2 (Chinn et al. [1]). *If* $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$ *and* $p = n_1 + n_2 + \cdots + n_k$, *then*

$$B(K_{n_1,n_2,\ldots,n_k}) = p - \lceil (n_1+1)/2 \rceil.$$

Proposition 3 (Chinn et al. [1]). *If* T *is a tree, then* $B(T) \leqslant |V(T)|/2$ *with equality if and only if* T *is a star of even order.*

Proposition 4 (Harper [10]). *For any proper numbering f of a graph of order n,*

$$\sum_{uv\in E(G)} |f(u)-f(v)| = \sum_{l=1}^{n} e(S_l),$$

where $S_l = \{v \in V(G) \mid f(v) \leqslant l\}$ *and* $e(S_l)$ *is the number of edges with one endvertex in* S_l *and the other in* $V(G) - S_l$.

2. Bandwidth on the composition of two graphs

Theorem 1. *Let G be a graph of order m and $n \geqslant 2$. If $B(G) \leqslant \lceil m/2 \rceil - 1$, then $B(K_n[G]) = mn - \lceil (m+1)/2 \rceil$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. Let $g: V(G) \to \{1, 2, \ldots, m\}$ be a proper numbering of G such that $B_g(G) = B(G)$. For set of integers $\{y_1, y_2, \ldots, y_m\}$, assume that

$$y_i = \begin{cases} i & \text{if } 1 \leqslant i \leqslant \lceil m/2 \rceil, \\ mn - m + i & \text{for } \lceil m/2 \rceil + 1 \leqslant i \leqslant m. \end{cases}$$

Then $B_f(K_n[G]) \leqslant mn - \lceil (m+1)/2 \rceil$ for any proper numbering f of $K_n[G]$ such that $f(u_1, v_j) = y_i$ if and only if $g(v_j) = i$, since (u_1, v_j) is not adjacent to (u_1, v_k) if $|g(v_j) - g(v_k)| \geqslant \lceil m/2 \rceil$. Therefore $B(K_n[G]) \leqslant mn - \lceil (m+1)/2 \rceil$. On the other hand, since $K_n[G] \supseteq K_{n(m)}$, the complete n partite graph with each partite set having m vertices, it follows from Proposition 2 that $B(K_n[G]) \geqslant B(K_{n(m)}) = mn - \lceil (m+1)/2 \rceil$. Thus $B(K_n[G]) = mn - \lceil (m+1)/2 \rceil$. $\square$

Theorem 2. *Let G be a graph of even order m and $n \geqslant 2$. If $B(G) = m/2$, then*

$$B(K_n[G]) = mn - \lceil (m+1)/2 \rceil + 1.$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. As in the first half of the proof of Theorem 2 we have $B(K_n[G]) \leqslant mn - \lceil (m+1)/2 \rceil + 1$. Suppose, to the contrary, that $B(K_n[G]) \neq mn - \lceil (m+1)/2 \rceil + 1$. Then $B(K_n[G]) \leqslant mn - \lceil (m+1)/2 \rceil$. Now, let f be a proper numbering of $K_n[G]$ such that $B_f(K_n[G]) = B(K_n[G])$. Let $w \in V(K_n[G])$ such that $f(w) = 1$. Without loss of generality, we assume that $w \in R_1 = \{(u_1, v_j)\colon 1 \leqslant j \leqslant m\}$. Since $B(K_n[G]) \leqslant mn - \lceil (m+1)/2 \rceil$, the vertices in R_1 are assigned the numbers $1, 2, \ldots, m/2, mn - m/2 + 1, mn - m/2 + 2, \ldots, mn$. Since R_1 induces the subgraph isomorphic to G, $B_f(K_n[G]) \geqslant mn - m/2$ if $B_g(G) \geqslant mn - m/2$ for any 1–1 mapping $g: V(G) \to \{1, 2, \ldots, m/2, mn - m/2 + 1, \ldots, mn\}$. In fact, if $B_g(G) < mn - m/2$ for a 1–1 mapping $g: V(G) \to \{1, 2, \ldots, m/2, mn - m/2 + 1, \ldots, mn\}$, then g induces a proper numbering $g_1: V(G) \to \{1, 2, \ldots, m\}$ as follows: for any $v_i \in V(G)$, $g_1(v_i) = g(v_i)$ if and only if $g(v_i) \leqslant m/2$ and $g_1(v_i) = g(v_i) - (n-1)m$ if and only if $g(v_i) > m/2$. Then it is easy to see that $B_{g1}(G) < m/2$, contradicting the hypothesis that $B(G) = m/2$. Thus $B_f(K_n[G]) \geqslant mn - m/2$, violating the assumption that $B(K_n[G]) \leqslant mn - \lceil (m+1)/2 \rceil$. Therefore $B(K_n[G]) = mn - \lceil (m+1)/2 \rceil + 1$. $\square$

The following corollaries follow directly from Theorems 1 and 2.

Corollary 1.

$$B(K_n[P_m]) = \begin{cases} mn - 1 & \textit{if } m \leqslant 2, \\ mn - \lceil (m+1)/2 \rceil & \textit{otherwise}. \end{cases}$$

Corollary 2. *If T is a tree of order m, then*

$$B(K_n[T]) = \begin{cases} mn - \lceil (m+1)/2 \rceil + 1 & \text{if } T \text{ is a star of even order,} \\ mn - \lceil (m+1)/2 \rceil & \text{otherwise.} \end{cases}$$

Corollary 3.

$$B(K_n[C_m]) = \begin{cases} mn - 1 & \text{if } m = 3, \\ mn - 2 & \text{if } m = 4, \\ mn - \lceil (m+1)/2 \rceil & \text{if } m \geqslant 5. \end{cases}$$

Corollary 4. *If* $\max\{n_1, n_2, \ldots, n_k\} \geqslant 5$ *and* $n \geqslant 2$, *then*

$$B(K_n[C_{n_1} \times C_{n_2} \times \cdots \times C_{n_k}]) = n \cdot \prod_{i=1}^{k} n_i - \left\lceil \left(\prod_{i=1}^{k} n_i + 1 \right) / 2 \right\rceil.$$

Theorem 3. *If* $\min\{n, k, n_1, n_2, \ldots, n_k\} \geqslant 2$, *then*

$$B(K_n[P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}])$$
$$= \begin{cases} n \cdot \prod_{i=1}^{k} n_i - \lceil (\prod_{i=1}^{k} n_i + 1)/2 \rceil + 1 & \text{if each } n_i = 2 \text{ and } k \leqslant 3, \\ n \cdot \prod_{i=1}^{k} n_i - \lceil (\prod_{i=1}^{k} n_i + 1)/2 \rceil & \text{otherwise.} \end{cases}$$

Proof. If $\max\{n_i\colon 1 \leqslant i \leqslant k\} > 2$, then it follows from Proposition 1 that $B(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) < \lceil \prod_{i=1}^{k} n_i/2 \rceil$. By Theorem 1,

$$B(K_n[P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}]) = n \cdot \prod_{i=1}^{k} n_i - \left\lceil \left(\prod_{i=1}^{k} n_i + 1 \right) \Big/ 2 \right\rceil.$$

Assume that $n_1 = n_2 = \cdots = n_k = 2$, then $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k} = P_2 \times P_2 \times \cdots \times P_2$ is isomorphic to the k-cube Q_k. Since

$$B(Q_k) = \sum_{i=0}^{k-1} \binom{i}{\lfloor i/2 \rfloor}$$

(see [11]), $B(Q_k) = |V(Q_k)|/2$ for $2 \leqslant k \leqslant 3$ and $B(Q_k) < |V(Q_k)|/2$ for $k \geqslant 4$. Our result then follows immediately from Theorems 1 and 2. □

Theorem 4. *For any graph G of order m,*

$$B(K_{1,n}[G]) = \lfloor ((n+2)m - 1)/2 \rfloor.$$

Proof. Let $V(K_{1,n}) = \{u_1, u_2, \ldots, u_{n+1}\}$ with u_{n+1} being the center and $V(G) = \{v_1, v_2, \ldots, v_m\}$. Let $S_i = \{(u_i, v_j)\colon 1 \leqslant j \leqslant m\}$. Then it follows from the definition for the composition of two graphs that each S_i induces a graph isomorphic to G and $K_{1,n}[G]$

contains a subgraph H isomorphic to $K_{m,nm}$ with two partite sets S_{n+1} and $\bigcup_{i=1}^{n} S_i$. By Proposition 2,

$$B(K_{1,n}[G]) \geqslant B(H) = (m + mn) - \lceil (mn + 1)/2 \rceil = \lfloor ((n+2)m - 1)/2 \rfloor.$$

To see that $B(K_{1,n}[G]) \leqslant \lfloor ((n+2)m - 1)/2 \rfloor$ consider the proper numbering f for $K_{1,n}[G]$ defined as follows:

Case 1. *n is even*:
$f(u_i, v_j) = (i-1)m + j$ for $1 \leqslant i \leqslant n/2$ and $1 \leqslant j \leqslant m$;
$f(u_{n+1}, v_j) = mn/2 + j$ for $1 \leqslant j \leqslant m$;
$f(u_i, v_j) = im + j$ for $n/2 + 1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Case 2. *n is odd*:
$f(u_i, v_j) = (i-1)m + j$ for $1 \leqslant i \leqslant (n-1)/2$ and $1 \leqslant j \leqslant m$;
$f(u_{(n+1)/2}, v_j) = (n-1)m/2 + j$ for $1 \leqslant j \leqslant \lceil m/2 \rceil$;
$f(u_{n+1}, v_j) = (n-1)m/2 + \lceil m/2 \rceil + j$ for $1 \leqslant j \leqslant m$;
$f(u_{(n+1)/2}, v_j) = (n-1)m/2 + j$ for $\lceil m/2 \rceil + 1 \leqslant j \leqslant m$;
$f(u_i, v_j) = im + j$ for $(n+3)/2 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Then $B_f(K_{1,n}[G]) = \lfloor ((n+2)m - 1)/2 \rfloor$. Therefore

$$B(K_{1,n}[G]) \leqslant \lfloor ((n+2)m - 1)/2 \rfloor.$$

Combining the two inequalities, we have

$$B(K_{1,n}[G]) = \lfloor ((n+2)m - 1)/2 \rfloor. \qquad \square$$

3. Edgesum for the composition of two graphs

Given a graph G and an arbitrary set $S \subseteq V(G)$, let $e(S)$ be the number of edges of G with one endvertex in S and the other in $V(G) - S$. For a proper numbering f of G, we denote $S_l(f) = \{v \in V(G): f(G) \leqslant l\}$ for $1 \leqslant l \leqslant |V(G)|$. Let G and H be two graphs of order n and m, respectively. We may assume $V(G[H]) = \{(i,j): 1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m\}$. The ith row of $G[H]$ is defined to be $R_i = \{(i,j): 1 \leqslant j \leqslant m\}$ for $1 \leqslant i \leqslant n$ and the jth column is $C_j = \{(i,j): 1 \leqslant i \leqslant n\}$ for $1 \leqslant j \leqslant m$. For any subset $S \subseteq V(G[H])$, let $a_i = |S \cap R_i|$ and $b_j = |S \cap C_j|$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Following the notation of [5], we define the *west shift* of S to be $\{(i,j) \in V(G[H]): j \leqslant a_i\}$and the *north shift* of S is $\{(i,j) \in V(G[H]): i \leqslant b_j\}$. A subset of $G[H]$ is *shift – invariant* if it is identical to both its west shift and north shift.

Remark 1 (Chvátalová [5]). *If S is a shift-invariant subset of $V(G[H])$, then $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_m$.*

Let $P_m = 1, 2, \ldots, m$ be a path of length $m-1$ and $V(K_n) = \{1, 2, \ldots, n\}$. We now develop an optimal numbering for the edgesum of $K_n[P_m]$.

Lemma 1. *For any subset $S \subseteq V(K_n[P_m])$, there is a shift-invariant subset S' such that $|S'| = |S|$ and $e(S') \leqslant e(S)$.*

Proof. Let $S \subseteq V(K_n[P_m])$, $a_i = |S \cap R_i|$ and $b_j = |S \cap C_j|$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. First, we show that if S^* is the west shift of S, then $e(S^*) \leqslant e(S)$. Let $e_j(A) =$ the number of edges between $A \cap R_j$ and $V(G) - A$ for $1 \leqslant i \leqslant n$, then $e(A) = \sum_{j=1}^{n} e_j(A)$. Then

$$e_j(S^*) = \begin{cases} \sum_{i \neq j} a_j(n - a_i) + 1 & \text{for } 0 < a_j < m, \\ \sum_{i \neq j} a_j(n - a_i) & \text{for } a_j = 0 \text{ or } a_j = m, \end{cases}$$

which is less than or equal to $e_j(S)$ for each j between 1 and n, $e(S^*) \leqslant e(S)$. Next, we claim that if S' is the north shift of S^*, then $e(S') = e(S^*)$. In fact, let $x_i = |S' \cap R_i|$ for $1 \leqslant i \leqslant n$, then $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$. Furthermore, there exists a permutation π of $(1, 2, \ldots, n)$ such that $x_i = a_{\pi(i)}$ for $1 \leqslant i \leqslant n$. It follows that

$$\begin{aligned} e(S') &= \sum_{j=1}^{n} e_j(S') = \sum_{j=1}^{n} \sum_{i \neq j} x_j(n - x_i) + t \\ &= \sum_{j=1}^{n} \sum_{i \neq j} a_j(n - a_i) + t = \sum_{j=1}^{n} e_j(S^*) = e(S^*), \end{aligned}$$

where $t = |\{a_i\colon 0 < a_i < m\}|$. Notice that S' is shift-invariant, $|S'| = |S^*| = |S|$, and $e(S') = e(S^*) \leqslant e(S)$, therefore the lemma follows. □

Lemma 2. *For any l between 1 and mn, let $l = kn + r$, where $0 \leqslant r < n$. If S is a shift-invariant l-subset of $V(G)$ such that $e(S) = \min\{e(A)\}$ over all l-subsets A of $V(G)$, then $b_1 = b_2 = \cdots = b_k = n$ and $b_{k+1} = r$.*

Proof. Let S be a shift-invariant l-subset of $V(G)$ such that $e(S) = \min\{e(A)\}$ over all l-subsets A of $V(G)$. Suppose, to the contrary, that $b_k < n$. Then there exists an integer t with $0 \leqslant t < k$ such that $b_t = n$ and $b_{t+1} < n$. Assume q is an integer such that $b_q > 0$ and $b_{q+1} = 0$, then $t + 2 \leqslant q \leqslant m$. It follows that $a_{b_{t+1}+1} = t$ and $a_{b_q} = q$, so $a_{b_{t+1}+1} + 2 \leqslant a_{b_q}$. Now, let $S' = (S - \{(b_q, q)\}) \cup \{(b_{t+1} + 1, t + 1)\}$. Then

$$\begin{aligned} e(S') &\leqslant e(S) + (a_1 + a_2 + \cdots + a_{b_q - 1} + a_{b_q + 1} + \cdots + a_n) \\ &\quad - (a_1 + a_2 + \cdots + a_{b_{t+1}} + a_{b_{t+1}+2} + \cdots + a_n) + 1 \\ &= e(S) + a_{b_{t+1}+1} - a_{b_q} + 1 < e(S), \end{aligned}$$

contradicting the choice for S. Therefore $b_k = n$, that is $b_1 = b_2 = \cdots = b_k = n$. Similarly, one can show that $b_{k+1} = r$. □

Theorem 5. *The numbering f such that*

$$f(i,j) = (j - 1)n + i \quad \textit{for } (i,j) \in V(K_n[P_m])$$

is an optimal numbering for the edgesum of $K_n[P_m]$.

Theorem 5 follows from Lemmas 1 and 2 together with Proposition 4.

Remark 2. The edgesum for K_n is

$$s(K_n) = \binom{n+1}{3}.$$

Remark 2 follows from the recurrence relation $s(K_n) = s(K_{n-1}) + n(n-1)/2$.

Corollary 5. *The edgesum for $K_n[P_m]$ is*

$$s(K_n[P_m]) = \binom{mn+1}{3} - n^2 . \binom{m+1}{3} + mn^2 - n^2.$$

Proof. Clearly,

$$\sum_{uv \in E(G)} |f(u) - f(v)| + \sum_{uv \in E(\bar{G})} |f(u) - f(v)| = \sum_{uv \in E(K_n)} |f(u) - f(v)|$$

for any graph G of order n and any proper numbering f of G. By Theorem 5, we have

$$\begin{aligned} s(K_n[P_m]) &= s(K_{mn}) - n^2 \cdot (s(K_m) - m + 1) \\ &= \binom{mn+1}{3} - n^2 \cdot \binom{m+1}{3} + mn^2 - n^2. \qquad \square \end{aligned}$$

Theorem 6 may be derived in a manner similar to that of Theorem 5.

Theorem 6. *The numbering f such that*

$$f(i,j) = (j-1)n + i \quad \textit{for } (i,j) \in V(K_n[C_m])$$

is an optimal numbering for the edgesum of $K_n[C_m]$.

References

[1] P.Z. Chinn, J. Chvátalová, A.K. Dewdney and N.E. Gibbs, The bandwidth problem for graphs and matrices—a survey, J. Graph Theory 6 (1982) 223–254.

[2] P.Z. Chinn, L. Yixun, Y. Jinjiang and K. Williams, Bandwidth of the composition of certain graph powers, Ars Combin. to appear.

[3] F.R.K. Chung, On optimal linear arrangements of trees, Comput. Math. Appl. 10 (1984) 43–60.

[4] F.R.K. Chung, Labelings of graphs, Selected topics in Graph Theory 3 (1988) 151–168.

[5] J. Chvátalová, Optimal labelling of a product of two paths, Discrete Math. 11 (1975) 249–253.

[6] J. Chvátalová, On the bandwidth problem for graphs, Ph.D. Thesis, Department of Combinatorics and Optimization, University of Waterloo, 1980.

[7] M.R. Garey, R.L. Graham, D.S. Johnson and D.E. Knuth, Complexity results for bandwidth minimization, SIAM J. Appl. Math., (1978) 477–495.

[8] M. Garey and D. Johnson, Computers and Intractability (Freeman, New York, 1979).

[9] M.R. Garey, D.S. Johnson and R.L. Stockmeyer, Some simplified NP-complete problems, Proc. 6th ACM Symp. on Theory of Computing (1974) 47–63.

[10] L.H. Harper, Optimal assignments of numbers to vertices, J Soc. Indust. Appl. Math. 12 (1) (1964).
[11] L.H. Harper, Optimal numberings and isoperimetric problems on graphs, J. Combin. Theory (1966) 385–393.
[12] L.H. Harper, Stabilization and the edgesum problem, Ars Combin. 4 (1977) 225–270.
[13] H. Lindsey II, Assignment of numbers to vertices, Amer. Math. Monthly 71 (1964) 508–516.
[14] J. Liu, On bandwidth sum for the composition of paths and cycles, Technical Report #TR/92-06, Department of Computer Science, Western Michigan University, 1992.
[15] J. Liu, Y.L. Lai and K. Williams, Bandwidth for the sum of k graphs, Ars Combin. 37 (1994) 149–155.
[16] J. Liu, J. Wang and K. Williams, Bandwidth for the sum of two graphs, Congr. Numer. 82 (1991) 79–85.
[17] G. Mitchison and R. Durbin, Optimal numberings of an $n \times n$ array, SIAM J. Algebraic Discrete Methods. 7 (1986) 571–582.
[18] C.H. Papadimitriou, The NP-completeness of the bandwidth minimization problem, Computing 16 (1976) 263–270.
[19] K. Williams, Determining bandwidth sum for certain graph sums, Congr. Numer. 90 (1992) 77–86.